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PRESSURE WAVES GENERATED BY ADDITION OF HEAT IN A GASEOUS MEDIUM

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PRESSURE WAVES GENERATED BY ADDITION OF HEAT IN A GASEOUS MEDIUM

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SUMMARY

The approximate formula of a linearized solution for the pressure field generated by a moderate rate of heat release is given. The analogies between the pressure waves generated by heat release and those generated by (1) mass release, (2) piston motion, or (3) a two-dimensional body in a supersonic stream are established analytically. The exact solution of an idealized problem in which a finite amount of heat is released uniformly at a section of a tube with a given rate, large or small, is also constructed. Though this idealized setup can be only approximately fulfilled in practice, the analysis does give an answer to a fundamental question: Given the rate of heat release at a section of a tube, how strong is the shock wave generated? A similar analysis is made for the pressure waves generated by a point source in three dimensions. Some applications of the theory are given.

INTRODUCTION

One of the basic problems in combustion aerodynamics is: What are the dynamic effects produced in a medium as a result of heat release in the medium? In particular, how strong are the pressure waves generated by heat release and to what extent are they important in a specific problem? Such questions arise naturally in the study of the spreading of autoignition, the transient development of detonation wave, ignition by compression, and many other time-dependent problems. The problem of estimating the pressure generated by heat release also occurs in other fields, for example, in the study of spark discharge and thermally driven acoustical oscillation as well as of the mechanical effects ("blast") produced in an atomic explosion (ref. 1).

The basic mechanism by which pressure waves are produced by heat addition is simply this: When heat is added to a volume of gas, the density of the gas is, in general, reduced. This causes an expansion of the volume occupied by the heated gas. The expansion of this volume produces the pressure waves.

To see this in a more quantitative manner, it is necessary to write down the fundamental equations governing the motion of the gas. Let p, ρ, T, and $\vec{u}$ be, respectively, the pressure, density, temperature and velocity vector of the flow, all being some functions of the position vector $\vec{r}$ and time t. Let R and C_v be the gas constant and specific heat at constant volume of the gas, both of which will be assumed constant in this preliminary study. The equations of continuity, momentum, and energy are

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \vec{u} = 0 \tag{1a}$$

$$\frac{D\vec{u}}{Dt} + \frac{1}{\rho}\nabla p = 0 \tag{1b}$$

$$C_v \frac{D}{Dt}\left(\log_e \frac{p}{\rho^{\gamma}}\right) = \frac{q\left(\vec{r},t\right)}{T} \tag{1c}$$

where $\frac{D}{Dt}$ represents the Stokes' derivative and $q\left(\vec{r},t\right)$ is the rate of heat release per unit mass of the medium at $\vec{r}$ and at the instant t. When heat is added into the medium from an external source, q may be considered as given and equations (1a), (1b), and (1c) together with the equation of state

$$p = \rho RT \tag{1d}$$

form a system of four equations which govern the four unknowns p, ρ, T, and $\vec{u}$.

Now it is observed that the rate of expansion of a given volume v of gas considered as a free body is measured by $\int_s \vec{u} \cdot \vec{n}\, ds$ where s is the bounding surface of the volume and $\vec{n}$ is the outward-drawn normal. Hence, by Gauss' theorem, it is also equal to

$$\int_v \nabla \cdot \vec{u}\, dv$$

It then follows immediately from continuity equation (1a) that whenever there is a change of density $\left(\frac{1}{\rho}\frac{D\rho}{Dt}\right)$ there will be a change of the volume. Now, when heat is added to the medium, there will, in general, be a change of pressure and density in accordance with energy equation (1c). (The relative amount of pressure change and density change resulting from heat addition can in principle always be determined in any specific problem by solving equations (1) jointly.) The change in density produces a change in volume occupied by the heated gas, which in turn generates the pressure waves.

It should be remarked that when heat is not added into the medium from external sources but is released by the fluid particles themselves the rate of heat release q must be considered as unknown and an additional equation is required to describe its variation. In combustion, q is usually given as a function of the local temperature. In other cases, the rate of heat release of the fluid particles is specified. In any event, the system of equations (1) will then be considerably more complicated than it appears to be.

Finally, one notes that the term $C_v \log_e\left(\frac{p}{\rho^\gamma}\right)$ in equation (1c) is directly related to the entropy of the gas by the formula

$$S - S_o = C_v \log_e\left[\left(\frac{p}{p_o}\right)\left(\frac{\rho_o}{\rho}\right)^\gamma\right] \tag{2}$$

where the subscript o denotes some reference state and S is the entropy of the gas.

For a moderate rate of heat release, the system of equations (1) can be linearized. The linearized theory is given in the following section where a "reduction theorem" is derived which enables one to reduce the problem of heat release in a tube to an equivalent problem of piston motion. An application of this reduction theorem to the one-dimensional case is given in the section entitled "APPLICATION OF REDUCTION THEOREM." When the rate of heat release is high, the linearized theory is no longer valid. In the section entitled "EXACT SOLUTION," an exact solution is obtained for the case when heat is released uniformly at a section of the tube at a constant rate (large or small). The corresponding problem in three dimensions is given in the section entitled "THREE-DIMENSIONAL THEORY." Some applications of the theory are also given.

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SYMBOLS

A	cross-sectional area of a tube
a	velocity of sound
C_v, C_p	specific heats at constant volume and constant pressure, respectively
E	energy per unit volume
F, F_1, F_2, F_3, G	some arbitrary functions
$H(x), H(t)$	step function
L	a characteristic length
M	Mach number
$m(\vec{r},t)$	mass injected per unit volume per unit time
$\vec{n}$	normal to a control surface
p	pressure
Q	rate of heat release, energy/sec
Q_o	constant rate of heat release (eqs. (46))
$\bar{Q}$	heating value of a mixture, energy/mass
$q(\vec{r},t)$, $q(x,y,z,t)$	rate of heat release per unit mass, energy/mass/sec
R	gas constant of a mixture
r	radial distance
r'	distance between field point and source point

$\vec{r}$	radius vector
r_c	radius of contact surface
r_1	radius of spherical shock wave
S	entropy
S_t	transformation velocity (i.e., flame speed)
s	bounding surface of volume v
T	temperature
T^*	temperature tending to ∞ in such manner that $\rho^* T^* =$ Constant
t	time
U	superficial velocity of a piston (see eq. (15))
u	velocity (in one-dimensional case)
$\vec{u}$	velocity vector
u_c	velocity of contact surface
u_r	radial velocity (in three-dimensional case)
u_1	velocity of flow immediately behind shock
V_s	velocity of propagation of shock wave
v	volume enclosed by control surface
x,y,z	Cartesian coordinate of a point
α	rate constant in parabolic heat-release law, equation (47), energy/sec^3

$\gamma = C_p/C_v$

δ	increment of or change of
$\delta(x)$	δ-function in one dimension
$\delta(x,y,z)$	δ-function in three dimensions
ϵ	any number, value of x
$\vec{\nu}$	vector normal to side of tube
ξ,η,ζ	coordinates of source point
ρ	density
ρ^*	density tending to zero in such a manner that $\rho^* T^*$ = Constant
ω	rate of heat release per unit area, energy/area/sec
ω_o	constant rate of heat release per unit area (eq. (20))
$\omega(y,z,t)$	rate of heat release at $x = \xi$, energy/area/sec

Subscripts:

c	state at contact surface
o	undisturbed state (i.e., that ahead of shock wave) except in case of variables representing rate of heat release ω and Q (see list of main symbols)
1	state immediately behind shock wave (which is also that immediately ahead of flame front in one-dimensional case)
2	state immediately behind flame front

LINEARIZED THEORY AND SOME ANALOGIES

Consider a medium in a uniform state with pressure p_o, density ρ_o, and temperature T_o. After the heat is added, the pressure, density, temperature, and velocity induced can be written as

$$p = p_o + \delta p \tag{3a}$$

$$\rho = \rho_o + \delta\rho \tag{3b}$$

$$T = T_o + \delta T \tag{3c}$$

$$\vec{u} = \delta\vec{u} \tag{3d}$$

If the rate of heat release is not too high, $\frac{\delta p}{p_o}$, $\frac{\delta\rho}{\rho_o}$, $\frac{\delta T}{T_o}$, and $\frac{|\delta\vec{u}|}{a_o} \ll 1$ where a_o is the speed of sound in the undisturbed medium. Equations (1) can then be linearized and

$$\frac{\partial}{\partial t}\left(\frac{\delta\rho}{\rho_o}\right) + \nabla \cdot \delta\vec{u} = 0 \tag{4a}$$

$$\frac{\partial}{\partial t}\left(\delta\vec{u}\right) + a_o^2 \nabla\left(\frac{\delta p}{\gamma p_o}\right) = 0 \tag{4b}$$

$$\frac{\partial}{\partial t}\left(\frac{\delta p}{\gamma p_o}\right) - \frac{\partial}{\partial t}\left(\frac{\delta\rho}{\rho_o}\right) = \frac{q\left(\vec{r},t\right)}{C_p T_o} \tag{4c}$$

$$\frac{\delta p}{p_o} + \frac{\delta\rho}{\rho_o} = \frac{\delta T}{T_o} \tag{4d}$$

Eliminate $\delta\rho/\rho_o$ from equations (4a) and (4c):

$$\frac{\partial}{\partial t}\left(\frac{\delta p}{\gamma p_o}\right) + \nabla \cdot \delta\vec{u} = \frac{q}{C_p t_o} \tag{5}$$

The equations governing the pressure and velocity fields are then given by equations (4b) and (5). Once the pressure field is known, the density and temperature field can be determined from equations (4c) and (4d) and the entropy spottiness from the linearized form of equation (2),

$$\frac{\delta S}{C_v} = \frac{\delta p}{p_o} - \gamma\,\frac{\delta\rho}{\rho_o} \tag{6}$$

It is now clear that "moderate rate of heat release" means $\frac{qL}{C_p T_o a_o} \ll 1$ where L is some characteristic length in the problem, for example, a length characterizing the dimension of the heating zone.

It is interesting to note that as far as the pressure and velocity fields are concerned, the governing equations (eqs. (5) and (4b)) are of the same form - in the linearized approximation - as those which govern the pressure and velocity fields produced by injecting fluid (isentropically) into the medium. For, in the latter case, the momentum equation in the linearized form is the same as equation (4b) while the linearized continuity equation is given by

$$\frac{\partial}{\partial t}\left(\frac{\delta p}{\gamma p_o}\right) + \nabla \cdot \delta\vec{u} = \frac{m(\vec{r},t)}{\rho_o} \tag{7}$$

where m is the rate at which fluid (measured in mass/volume/second) is introduced into the medium. In fact, all one has to do to get the same pressure and velocity fields for the two cases is to match the parameter q/C_pT_o with m/ρ_o. This suggests that the noise generated in a nonuniform combustion (e.g., in turbulent combustion) will radiate as a source field when examined at a large distance away from the combustion region. (It is not known if any experiment has been carried out in the study of this phenomenon.)

If $\delta\vec{u}$ is eliminated from equations (5) and (4b) the pressure field is found to satisfy the wave equation

$$\frac{\partial^2}{\partial t^2}\left(\frac{\delta p}{\gamma p_o}\right) - a_o^2\, \nabla^2\left(\frac{\delta p}{\gamma p_o}\right) = \frac{\partial}{\partial t}\left(\frac{q}{C_pT_o}\right) \tag{8}$$

It is clear from equation (8) that the pressure produced by heat addition depends upon the rate of heat release, and whenever there is a change of rate of heat release there will be pressure waves generated, a fact which one should have expected in the first place.

Now consider the gases contained in a tube. The x-axis will be chosen parallel to the axis of the tube. The heat added to the medium $q(x,y,z,t)$ is assumed to be given for $t > 0$. For $t \leqq 0$, $q(x,y,z,t)$ is assumed to be identically zero. Now since

$$q(x,y,z,t) = \int_{-\infty}^{\infty} q(\xi,y,z,t)\ \delta(x - \xi)\ d\xi \tag{9}$$

and equation (8) is linear (so that the superposition principle is valid), it is clearly sufficient to analyze the effect of heat release when $q(x,y,z,t) = \frac{1}{\rho_o}\,\omega(y,z,t)\ \delta(x - \xi)$, that is, when heat is added only at the section $x = \xi$. The quantity $\omega(y,z,t)$ has a physical meaning of its own. It is the heat generated per unit area per unit time at the section $x = \xi$, for

$$\iint \omega(y,z,t)\ dy\ dz = \text{Total heat generated at } x = \xi \text{ in unit time}$$

where the integration is taken over the cross section of the tube at $x = \xi$. Mathematically, this amounts to finding the elementary solution of equation (8), for once the pressure field produced by $\omega(y,z,t)$ at $x = \xi$ is known, the general case of heat release by arbitrarily distributed sources $q(x,y,z,t)$ can be obtained by replacing the term $\omega(y,z,t)$ in the formulas found for the pressure and velocity field with $\rho_o q(\xi,y,z,t)\ d\xi$ and integrating them from $\xi = -\infty$ to ∞.

By a translation of the coordinate axes the plane at which heat is added can be made to be the plane $x = 0$. The pressure field produced by heat released at the plane $x = 0$ will then satisfy the differential equation

$$\frac{\partial^2}{\partial t^2}\left(\frac{\delta p}{\gamma p_o}\right) - a_o^2\ \nabla^2\left(\frac{\delta p}{\gamma p_o}\right) = \frac{\partial}{\partial t}\left[\frac{\omega(y,z,t)\ \delta(x)}{\rho_o C_p T_o}\right] \tag{10}$$

and the boundary condition

$$\frac{\partial}{\partial \nu}\left(\frac{\delta p}{\gamma p_o}\right) = 0 \tag{11}$$

at the wall of the tube. (Here $\vec{\nu}$ stands for the normal vector to the side of the tube.) Equation (11), according to equation (4b), is an equivalent statement of the requirement that there should be no flow across the wall of the tube. If it is further assumed that initially at $t = 0$ there are no disturbances inside the tube, there are also the initial conditions

$$\left(\frac{\delta p}{\gamma p_o}\right)_{t=0} = 0 \tag{12a}$$

$$\left[\frac{\partial}{\partial t}\left(\frac{\delta p}{\gamma p_o}\right)\right]_{t=0} = 0 \tag{12b}$$

The solution of the above problem can be facilitated with the aid of the "reduction theorem" given below.

Reduction Theorem

Let heat be released at a section $x = 0$ of a tube of constant cross section and of infinite length at a rate of $\omega(y,z,t)$ units of energy per unit area per unit time. The pressure and velocity fields generated as a result of this heat release are identical to those produced by two pistons at $x = 0t$ moving away from each other (along the axis of the tube) with the same superficial velocity $\frac{\gamma - 1}{2\gamma}\frac{\omega(y,z,t)}{p_o}$.

In this manner, the problem of pressure waves generated by heat release is reduced to a problem of piston motion. This reduction enables one to make use of the result of known theories (e.g., ref. 2) for the problem at hand. It is possible to generalize the theorem to a tube of, for example, finite length. This will be discussed later. The proof of the reduction theorem is very similar to that used in deriving the "impulse method" of solving a nonhomogeneous wave equation. (See, e.g., ref. 3.)

Proof of Reduction Theorem

First of all, observe that every solution of equation (10) will be a solution of the homogeneous wave equation at all points except $x = 0$, since $\delta(x) = 0$ for $x \neq 0$. Next perform an integration of equation (10) with respect to x from $-\epsilon$ to ϵ and then let $\epsilon \rightarrow 0$. If it is assumed that $\frac{\partial^2}{\partial t^2}\left(\frac{\delta p}{\gamma p_o}\right)$, $\frac{\partial^2}{\partial y^2}\left(\frac{\delta p}{\gamma p_o}\right)$, and $\frac{\partial^2}{\partial z^2}\left(\frac{\delta p}{\gamma p_o}\right)$ are bounded near $x = 0$, a condition which can certainly be satisfied if $\omega(y,z,t)$ is a function smooth enough in y, z, and t, equation (10) becomes

$$-a_o^2\left(\frac{\partial}{\partial x}\frac{\delta p}{\gamma p_o}\right)_{x=0+} + a_o^2\left(\frac{\partial}{\partial x}\frac{\delta p}{\gamma p_o}\right)_{x=0-} = \frac{\partial}{\partial t}\left[\frac{\omega(y,z,t)}{\rho_o C_p T_o}\right] \tag{13}$$

By symmetry, $\delta p/\gamma p_o$ is an even function of x so that it is sufficient to consider the pressure field in the region $x > 0$ only. Also, by symmetry, $\left(\frac{\partial}{\partial x}\frac{\delta p}{\gamma p_o}\right)_{x=0-} = -\left(\frac{\partial}{\partial x}\frac{\delta p}{\gamma p_o}\right)_{x=0+}$ so that equation (13) can be written simply as

$$\left(\frac{\partial}{\partial x}\frac{\delta p}{\gamma p_o}\right)_{x=0+} = -\frac{1}{2a_o^2}\frac{\partial}{\partial t}\left[\frac{\gamma - 1}{\gamma}\frac{\omega(y,z,t)}{p_o}\right] \tag{13a}$$

In other words, if $\delta p/\gamma p_o$ is a solution of equation (10), it must also satisfy condition (13a) as $x \to 0+$. Consequently, the solution $\delta p/\gamma p_o$ of equation (10) satisfying conditions (11) and (12) must satisfy the homogeneous wave equation

$$\frac{\partial^2}{\partial t^2}\left(\frac{\delta p}{\gamma p_o}\right) - a_o{}^2 \nabla^2\left(\frac{\delta p}{\gamma p_o}\right) = 0 \tag{14}$$

for $x > 0$ as well as conditions (11), (12), and (13a). But the pressure field in the region $x > 0$ produced by a piston at $x = 0+$ moving with a superficial velocity

$$U(y,z,t) = \frac{\gamma - 1}{2\gamma} \frac{\omega(y,z,t)}{p_o} \tag{15}$$

in the x-direction will precisely satisfy equations (11), (12), (13a), and (14) for $x > 0$.[1] Moreover, since the pressure field generated by a moving piston is uniquely determined in the linearized theory, it is also the only solution which satisfies equations (11), (12), (13a), and (14). Consequently, the pressure field generated by heat release and piston motion must be identical. The velocity produced by heat release and by piston motion are also identical for $x > 0$, since in both cases the velocity and pressure are related by formula (4b) and

$$\frac{\partial}{\partial t}\left(\frac{\delta p}{\gamma p_o}\right) + \nabla \cdot \delta\vec{u} = 0$$

in this region. (Cf. eq. (5) and observe that $q(x,y,z,t) = \frac{1}{\rho_o}\omega(y,z,t)\delta(x)$ and $\delta(x) = 0$ for $x \neq 0$. Hence $q = 0$ for $x \neq 0$.) This completes the proof of the reduction theorem.

It is clear from the above proof that the same conclusion will hold for tubes with nonuniform cross sections which are symmetrical about the plane $x = 0$. It also holds for tubes of uniform section of finite length provided that it is limited to the region ahead of the wave front reflected from the end of the tube. For the most general case, in which no such restrictions are imposed, a similar but less simple analogy can be constructed. In this case the two pistons will be moving with different velocities, and they are so related that not

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[1]That it satisfies equation (13a) follows from the momentum equation in the x-direction (eq. (4b)), the boundary condition that there should be no flow crossing the face of the piston, and equation (15).

only is equation (13) satisfied but also the condition $\left(\frac{\delta p}{\gamma p_o}\right)_{x=0+} = \left(\frac{\delta p}{\gamma p_o}\right)_{x=0-}$ is fulfilled. (The equality of pressure is a direct consequence of momentum balance at the section $x = 0$ when all the second-order terms are neglected.) In the next section, application of this reduction theorem will be given.

The reduction theorem was originally formulated as a means of reducing a problem of heat addition to a problem of piston motion. However, the fictitious pistons described in the theorem actually have a real physical significance. It has been shown in the introduction that when heat conduction is neglected the flow field resulting from heat addition into a medium is really caused by the volumetric expansion of the hot gas resulting from the heating. In the case of heat addition at one plane (say, $x = 0$), the rate of this expansion can be calculated. In fact, the faces of the two fictitious pistons must be precisely the two interfaces which separate the cool gas from the heated gas, since they start from $x = 0\pm$ at $t = 0$ and move out with a speed equal to the speed of the fluid particle.

Dr. Harold Mirels pointed out that the reduction theorem states that the pressure and velocity fields produced by addition of ω units of energy per unit time per unit area are equivalent to those produced by two pistons at $x = 0\pm$ moving away from each other with velocity

$$\delta u = \frac{\gamma - 1}{2\gamma} \frac{\omega}{p_o}$$

However, the rate of work done by such pistons is $2p_o\delta u = \frac{\gamma - 1}{\gamma} \omega$, whereas the heat input is ω. Thus there are $\frac{1}{\gamma}\omega$ units of energy apparently unaccounted for. He went on to say that this is due to the step-function behavior of mass and energy at $x = 0$ (as can be deduced for the particular case discussed in the next section from equations (22) to (24)).

A more direct physical explanation of the energy that appears "unaccounted for" is the following. The ω units of heat energy per unit area per unit time added can be divided into two parts: That responsible for the setting up of the pressure and velocity fields (which, according to the reduction theorem, must be equal to $2p_o\delta u = \frac{\gamma - 1}{\gamma}\omega$), and that responsible for heating up the medium. On the other hand, in the case of the flow field produced by the piston motion, all the energy goes into the setting up of the pressure and velocity fields. The amount of energy that was previously unaccounted for must then be the thermal energy stored in the hot gas. This can be seen from the following calculation. From the reduction theorem, it is concluded that the flow fields produced by heating and piston motion are identical up to the faces of the pistons. In the case of heat addition, the hot gas filled the space between the faces of the two fictitious pistons. In a time interval δt, the increase in volume of the hot gas is $2A\delta u\delta t$, where A is the cross-sectional area of the tube. The internal energy stored in the hot gas is increased (during the same interval) by $2A\delta u\delta t(\rho_o + \delta\rho)C_v(T_o + \delta T)$, that is, by $\frac{2}{\gamma - 1} A\delta u\delta t(p_o + \delta p)$, or, to the order of the linearized approximation, simply by $\frac{2}{\gamma - 1} p_o\delta u\delta tA$. Hence the rate of increase in the internal energy stored in the hot gas per unit cross-sectional area at any instant t is $\frac{2}{\gamma - 1} p_o\delta u = \frac{2}{\gamma - 1} p_o\left(\frac{\gamma - 1}{2\gamma} \frac{\omega}{p_o}\right) = \frac{\omega_o}{\gamma}$, which is exactly equal to the amount of energy that appears unaccounted for if only the flow field produced by the fictitious pistons is examined. (The above calculation is made for the case ω = Constant across the cross-sectional area of the tube. For the more general case $\omega = \omega(y,z,t)$, the same calculation applies except that one should write the integral $\int . . . dA$ instead of A in the calculation.)

In the linearized theory presented here, only the effect of compressibility has been taken into account and the effect of heat conduction has been neglected completely. Linearized theory taking heat conductivity into account was recently investigated by Wu (ref. 4).

APPLICATION OF REDUCTION THEOREM

As an application of the reduction theorem, consider the case where

$$\omega(y,z,t) = \omega(t) \tag{16}$$

that is, the case in which heat is uniformly released at the plane $x = 0$. According to the reduction theorem, the pressure and velocity fields induced are the same as those generated by two pistons at $x = 0\pm$ moving away from each other with velocities

$$U(t) = \pm\frac{\gamma - 1}{2\gamma} \frac{\omega(t)}{p_o} \tag{17}$$

respectively. If the tube containing the gas is of constant cross section, the pressure and velocity fields produced by the pistons will be one dimensional, and they are given by the well-known formulas

$$\frac{\delta p}{\gamma p_o} = \frac{\gamma - 1}{2\gamma} \frac{\omega\left(t - \frac{x}{a_o}\right)}{a_o p_o} \qquad \text{for} \quad x > 0 \tag{18a}$$

$$\frac{\delta p}{\gamma p_o} = \frac{\gamma - 1}{2\gamma} \frac{\omega\left(t + \frac{x}{a_o}\right)}{a_o p_o} \qquad \text{for} \quad x < 0 \tag{18b}$$

$$\delta u = \frac{\gamma - 1}{2\gamma} \frac{\omega\left(t - \frac{x}{a_o}\right)}{p_o} \qquad \text{for} \quad x > 0 \tag{19a}$$

$$\delta u = -\frac{\gamma - 1}{2\gamma} \frac{\omega\left(t + \frac{x}{a_o}\right)}{p_o} \qquad \text{for} \quad x < 0 \tag{19b}$$

In particular, if the rate of heat release is constant for $t > 0$, that is,

$$\omega(t) = \left\{ \begin{array}{ll} 0 & \text{for} \quad t \leqq 0 \\ \omega_o & \text{for} \quad t > 0 \end{array} \right\} \tag{20}$$

or, written in terms of the step function, $\omega(t) = \omega_o H(t)$, the pressure field is given by

$$\frac{\delta p}{\gamma p_o} = \frac{\gamma - 1}{2\gamma} \frac{\omega_o}{a_o p_o} H\left(t - \frac{x}{a_o}\right) \qquad \text{for} \quad x > 0 \tag{21a}$$

$$\frac{\delta p}{\gamma p_o} = \frac{\gamma - 1}{2\gamma} \frac{\omega_o}{a_o p_o} H\left(t + \frac{x}{a_o}\right) \qquad \text{for} \quad x < 0 \tag{21b}$$

This shows that, when heat is released at a constant rate of ω_o units of energy per unit area per unit time into the medium, two compression waves of equal strength are generated and propagate away from the heating zone. These compression waves have a strength, measured in terms of the ratio of pressure jump δp across the wave to the undisturbed pressure p_o, equal to $\frac{\gamma - 1}{2} \frac{\omega_o}{a_o p_o}$. It is thus seen that the strength of the pressure waves produced by heat addition to the medium is usually small. (Thus, if $\omega_o = 10$ Btu per square foot per second,

a_o = 1,000 feet per second, and p_o = 2,000 pounds per square foot, then $\frac{\delta p}{p_o} = 0.2 \times \frac{10 \times 778}{1,000 \times 2,000} = 778 \times 10^{-6} = 0.078$ percent.) When the rate of heat release is high, the pressure waves generated may be quite intense. However, for a very high rate of heat release the above formulas can no longer be used. In fact, the formula is correct only if the nondimensional heat-release parameter $\frac{\omega_o}{a_o p_o} \ll 1$, since it is based on a linearized theory. However, the simplicity of the pressure and velocity field predicted from the linearized theory in the last instance leads one to suspect that an exact solution can be found by replacing the infinitesimal pressure steps by two shock waves. The main problem is again to determine the strength of the shock wave, given the rate of heat release $\omega_o/a_o p_o$. This problem will be considered in the next section.

It is seen from equations (18) and (19) that the pressure is continuous at $x = 0$, but the velocity is discontinuous there, since

$$(\delta u)_{x=0+} = \frac{\gamma - 1}{2\gamma} \frac{\omega(t)}{p_o}$$

$$(\delta u)_{x=0-} = -\frac{\gamma - 1}{2\gamma} \frac{\omega(t)}{p_o}$$

The fact that the flow is leaving the section $x = 0$ at both $x = 0+$ and $x = 0-$ leads one to inquire if all the conservation theorems are really satisfied at $x = 0$. (It is certainly obvious that they are satisfied at all values of $x \neq 0$.)

It will now be shown that these laws are indeed satisfied at $x = 0$. For this purpose it is necessary to calculate the density and temperature field. Integrating equation (4c) with respect to t, By equation (4d),

$$\frac{\delta T}{T_o} = \frac{\gamma - 1}{\gamma}\frac{\delta p}{p_o} + \int_0^t \frac{\omega(t)\delta(x)}{\rho_o C_p T_o}\,dx = \begin{cases} \dfrac{(\gamma - 1)^2}{2\gamma}\dfrac{\omega\left(t - \frac{x}{a_o}\right)}{a_o p_o} + \delta(x)\displaystyle\int_0^t \frac{\omega(t)}{\rho_o C_p T_o}\,dt & \text{for } x > 0 \\ \dfrac{(\gamma - 1)^2}{2\gamma}\dfrac{\omega\left(t + \frac{x}{a_o}\right)}{a_o p_o} + \delta(x)\displaystyle\int_0^t \frac{\omega(t)}{\rho_o C_p T_o}\,dt & \text{for } x < 0 \end{cases} \tag{23}$$

Consider a control surface formed by the planes $x = \epsilon$, $x = -\epsilon$ (where ϵ may be any number, large or small), and the segment of wall of the tube between these planes. Let A be the cross-sectional area of the tube. The conservation of mass and energy can be expressed as

$$-\frac{\partial}{\partial t}\left(\int_{-\epsilon}^{\epsilon} \rho A\,dx\right) = 2\rho(\epsilon,t)u(\epsilon,t)A \tag{24a}$$

$$-\frac{\partial}{\partial t}\left[\int_{-\epsilon}^{\epsilon} \rho\left(E + \frac{1}{2}u^2\right)A\,dx\right] = 2\rho(\epsilon,t)u(\epsilon,t)A\left(E + \frac{1}{2}u^2\right)_{x=\epsilon} + 2p(\epsilon,t)u(\epsilon,t)A - \int_{-\epsilon}^{\epsilon} \rho\left[\frac{\omega(t)}{\rho_o}\,\delta(x)\right]\,dx \tag{24b}$$

Substituting equations (3), (18), and (19) into the above equations and neglecting the second-order small quantities (in accordance with the linearized approximation), it is found that equations (24) are indeed satisfied for all values of ϵ. (By reason of symmetry of the flow field with respect to $x = 0$, the momentum equation formulated for the same control surface is automatically satisfied.) If ϵ is taken as an arbitrarily small number, this establishes the validity of the conservation laws at $x = 0$. If ϵ is chosen large enough to enclose the whole disturbed field, it establishes the validity of the over-all conservation laws for the system. Note that these conservation laws will not be satisfied if the pressure and velocity fields are related to the rate of heat release ω in any other manner than by equations (18) and (19).

There are two aspects of this analysis which deserve some criticism. First of all, it is clear from equation (23) that, in general, $\delta T/T_o$ is infinite at $x = 0$. This is, in fact, a direct consequence of the addition of a finite amount of heat in a plane. Since a "plane of gas" (instead of a volume) possesses no heat capacity, the temperature must become infinite. Now, if $\delta T/T_o$ is infinite, the linearization must break down at $x = 0$ so that the solution really does not represent any physical state of affairs at $x = 0$. Secondly, the density $\delta\rho/\rho_0$, according to equation (22), becomes $-\infty$ at $x = 0$. Here there is not only an infinity but also a negative density which is not even physically conceivable. It will be seen in the next section that the negative density is really a consequence of linearization.

Despite the absurdity of the behavior of the solution at $x = 0$,[2] all the other conclusions derived above, such as the dependence of the strength of the pressure wave on the rate of heat release (eq. (21)) are to be trusted because only this dependence will insure the conservation of energy, mass, and momentum locally at all points as well as for the whole system (i.e., over-all balance). This state of affairs will be clarified when an exact solution is constructed in the next section.

Other evidence which indirectly justifies the value of this solution is found when use is made of the above solution to get the pressure and velocity fields produced by heat sources $q(x,t)$ which are not distributed in one plane but over a region (volume) of the tube. According to the rule given (see eq. (9) and the paragraph following it), it is necessary only to replace x in formulas (18), (19), (22), and (23) by $x - \xi$ (i.e., perform a translation) and then replace the function ω in these formulas by $\rho_0 q(\xi,t)\, d\xi$, and, finally, integrate ξ from $-\infty$ to ∞. Thus, the pressure field is given by

$$\frac{\delta p}{\gamma p_0} = \frac{\gamma - 1}{2\gamma}\frac{1}{a_0 p_0}\left[\int_{-\infty}^{x} \rho_0 q\left(\xi, t - \frac{x - \xi}{a_0}\right) d\xi + \int_{x}^{\infty} \rho_0 q\left(\xi, t + \frac{x - \xi}{a_0}\right) d\xi\right]$$

But, since $q(\xi,t) \equiv 0$ for $t \leqq 0$, the last formula can also be written as

$$\frac{\delta p}{\gamma p_0} = \frac{1}{2a_0}\left[\int_{x-a_0 t}^{x} \frac{q\left(\xi, t - \frac{x - \xi}{a_0}\right)}{C_p T_0} d\xi + \int_{x}^{x+a_0 t} \frac{q\left(\xi, t + \frac{x - \xi}{a_0}\right)}{C_p T_0} d\xi\right] \tag{25}$$

which is well known as the correct solution of the one-dimensional-wave equation

$$\frac{\partial^2}{\partial t^2}\left(\frac{\delta p}{\gamma p_0}\right) - a_0^2 \frac{\partial^2}{\partial x^2}\left(\frac{\delta p}{\gamma p_0}\right) = \frac{\partial}{\partial t}\left[\frac{q(x,t)}{C_p T_0}\right]$$

subjected to the initial conditions $\left(\frac{\delta p}{\gamma p_0}\right)_{t=0} = 0$ and $\left[\frac{\partial}{\partial t}\left(\frac{\delta p}{\gamma p_0}\right)\right]_{t=0} = 0$

[2]To be sure, to begin with, it is physically impossible to add a finite amount of heat in a plane (instead of a volume) of gas. However, this is beside the point because this state of affairs can at least be approximated in practice.

Likewise, the velocity, density, and temperature fields are given by

$$\delta u = \frac{1}{2}\int_{x-a_ot}^{x} \frac{q\left(\xi,t - \frac{x - \xi}{a_o}\right)}{C_pT_o}\,d\xi - \frac{1}{2}\int_{x}^{x+a_ot} \frac{q\left(\xi,t + \frac{x - \xi}{a_o}\right)}{C_pT_o}\,d\xi \tag{26}$$

$$\frac{\delta\rho}{\rho_o} = \frac{1}{2a_o}\left[\int_{x-a_ot}^{x} \frac{q\left(\xi,t - \frac{x - \xi}{a_o}\right)}{C_pT_o}\,d\xi + \int_{x}^{x+a_ot} \frac{q\left(\xi,t + \frac{x - \xi}{a_o}\right)}{C_pT_o}\,d\xi\right] - \int_0^t \frac{q(x,t)}{C_pT_o}\,dt \tag{27}$$

$$\frac{\delta T}{T_o} = \frac{\gamma - 1}{2a_o}\left[\int_{x-a_ot}^{x} \frac{q\left(\xi,t - \frac{x - \xi}{a_o}\right)}{C_pT_o}\,d\xi + \int_{x}^{x+a_ot} \frac{q\left(\xi,t + \frac{x - \xi}{a_o}\right)}{C_pT_o}\,d\xi\right] + \int_0^t \frac{q(x,t)}{C_pT_o}\,dt \tag{28}$$

Note that in the expressions for the temperature (eq. (28)) and density (eq. (27)) the "infinities," which were originally contained in expressions (22) and (23), disappear after superposition.

To get some idea as to the pressure and velocity distributions inside a narrow (but nonzero) band of heating zone, a calculation is made for the case

$$q(x,t) = \begin{cases} 0 & \text{for } t \leqq 0;\ \text{also for } t > 0 \text{ if } |x| > \epsilon \\ q_o & \text{for } t > 0 \text{ and } |x| < \epsilon \end{cases}$$

where ϵ is some small number. Typical pressure and velocity distributions for various times are shown in figure 1.

The solid lines in figure 1 are the pressure and velocity distributions. The dotted lines are the construction lines. Points between aa' do not realize the heating zone is cut off at $x = \pm\epsilon$ and think that it extends from $-\infty$ to ∞. By symmetry, therefore, the fluid particle tends to move toward neither the left nor the right. Hence in this region $\delta u = 0$, and the density, by the continuity equation, remains unchanged, while the pressure increases linearly with the temperature (constant-volume combustion). Points between ab do realize that the heating zone is cut off at $x = \epsilon$ but do not know if it is also cut off at $x = -\epsilon$. The pressure and velocity differ at different points in ab depending on how much each point knows about the conditions at each side of the point $x = \epsilon$. The same remarks hold for points between a'b'.

Before concluding the discussion of the reduction theorem, one should mention that when heat is released uniformly at one plane, that is, $\omega(y,z,t) = \omega(t)$, it is possible to construct a third analogy, namely, that between the pressure waves generated by heat release and a two-dimensional airfoil in supersonic flow.[3] For the reduction theorem shows that the pressure field produced by heat release satisfies the system of equations (14), (13), (12a), and (12b), which in the present case become

$$\frac{\partial^2}{\partial t^2}\left(\frac{\delta p}{\gamma p_o}\right) - a_o^2 \frac{\partial^2}{\partial x^2}\left(\frac{\delta p}{\gamma p_o}\right) = 0 \tag{29a}$$

$$\left[\frac{\partial}{\partial x}\left(\frac{\delta p}{\gamma p_o}\right)\right]_{x=0\pm} = \mp \frac{1}{a_o^2} \frac{\partial}{\partial t}\left[\frac{\gamma - 1}{2\gamma} \frac{\omega(t)}{p_o}\right] \tag{29b}$$

$$\left(\frac{\delta p}{\gamma p_o}\right)_{t=0} = 0 \tag{29c}$$

$$\left[\frac{\partial}{\partial t}\left(\frac{\delta p}{\gamma p_o}\right)\right]_{t=0} = 0 \tag{29d}$$

On the other hand, if a symmetrical two-dimensional airfoil $y = \pm f(x)$ in a uniform supersonic stream of Mach number M is considered, the pressure field produced by the airfoil satisfies the following set of equations:

$$\frac{\partial^2}{\partial x^2}\left(\frac{\delta p}{\gamma p_o}\right) - \frac{1}{M_o^2 - 1} \frac{\partial^2}{\partial y^2}\left(\frac{\delta p}{\gamma p_o}\right) = 0 \tag{30a}$$

$$\left[\frac{\partial}{\partial y}\left(\frac{\delta p}{\gamma p_o}\right)\right]_{y=0\pm} = \mp M_o^2 \frac{\partial}{\partial x} f(x) \tag{30b}$$

[3]The author is indebted to Dr. L. S. G. Kovásznay for suggesting this analogy in the early stage of this analysis.

$$\left(\frac{\delta p}{\gamma p_o}\right)_{x=0} = 0 \tag{30c}$$

$$\left[\frac{\partial}{\partial x}\left(\frac{\delta p}{\gamma p_o}\right)\right]_{x=0} = 0 \tag{30d}$$

(According to the momentum equation in the y-direction eq. (30b) is equivalent to the statement that the flow at the surface of the airfoil must be tangent to the airfoil. The last two conditions are equivalent to the statement that there should be no incoming waves in the flow field.)

A comparison of equations (29) and (30) shows that there exists an analogy between the pressure field developed by the uniform heat addition at a plane and the two-dimensional symmetrical airfoil in a supersonic stream. In fact, it is necessary only to imagine the time coordinate as the space coordinate in the direction of the supersonic stream and choose $M^2 = 1 + \frac{1}{a_o^2}$ and the shape of the airfoil according to the formula $f(x) = \frac{\gamma - 1}{2\gamma} \frac{1}{1 + a_o^2} \frac{\omega(x)}{p_o}$.

EXACT SOLUTION

Let heat be uniformly released at the section $x = 0$ of a tube of constant cross section and of infinite length at a constant rate of ω_o units of energy per unit area per unit time. (Hence, if the total heat released per second is Q and the cross-sectional area of the tube is A, $\omega_o = Q/A$.) When the nondimensional heat-release parameter $\omega_o/a_o p_o$ was small compared with unity, two compression waves of strength $\frac{\gamma - 1}{2} \frac{\omega_o}{a_o p_o}$, propagating away from each other with the velocity of sound a_o of the undisturbed medium, were found to be generated at $x = 0$ (see eq. (21)). When $\omega_o/a_o p_o$ is not small, equation (21) is no longer valid. Furthermore, it is expected that, instead of the two infinitesimal pressure steps, two shock waves would be generated. The question is again to determine the strength of the shock wave generated, given the rate of heat release per unit area ω_o.

The linearized solution (21), though invalid for a high rate of heat release, does suggest the following propositions:

(1) The shock waves generated at $x = 0$ are of equal strength and will maintain their strength as they propagate away from each other.

(2) The pressure between the two shock waves is uniform and equal to the pressure immediately behind the shocks.

(3) The velocity field behind the shock waves is discontinuous at $x = 0$ and is an odd function of x. It is uniform for the regions $x > 0$ and $x < 0$, respectively, and assumes the value of the velocity of the gas immediately behind the shock in the region $x > 0$ and $x < 0$, respectively.

In addition to these propositions, due account will be taken of the motion of the fluid particles which is neglected in the linearized theory. It would be expected that:

(4) Two contact surfaces which form the boundaries separating the hot and cool gas are generated at $x = 0$ at the instant when heat is first added to the medium and move away from each other with the velocity of the fluid particles.

(5) Since the hot gas originated from a single plane $x = 0$, the temperature of the gas between the two contact surfaces must tend to infinity.

(6) For the same reason, the density of the hot gas between the two contact surfaces must tend to zero in such a manner that $\rho T = \text{Constant}$ in accordance with the gas law $\rho T = p/R$ and proposition (2).

These propositions can be substantiated by a formal argument based on dimensional reasoning. The argument goes as follows: The undisturbed medium can be characterized by two of its thermodynamic state parameters, say, the pressure p_o and temperature T_o. Since the velocity of sound a_o in the undisturbed medium is uniquely related to the temperature T_o, p_o and a_o will be used as the two parameters characterizing the undisturbed medium. The strength of the shock wave can be described in terms of the pressure ratio p_1/p_o across the shock, where p_1 is the pressure immediately behind the shock. It is clear that, in general, the strength of the shock wave depends on the rate of heat release per unit area ω_o, the state of the undisturbed medium being characterized by p_o and a_o as well as by the time t. That is,

$$\frac{p_1}{p_o} = F(\omega_o, a_o, p_o, t) \tag{31a}$$

Since the viscous and heat-conductive effects have been neglected in this preliminary study, these variables do not enter into equation (31a). Also, for uniform heat release in a tube of constant cross section the flow field is one dimensional so that the dimension of the tube does not enter into the problem as a relevant characteristic length. Now equation (31a) must be dimensionally correct. But the four variables ω_o, p_o, a_o, and t can only be combined into a single nondimensional parameter, namely, $\frac{\omega_o}{a_o p_o}$, which does not contain the variable t. Consequently,

$$\frac{p_1}{p_o} = F_1\left(\frac{\omega_o}{a_o p_o}\right) \tag{31b}$$

that is, the shock strength must be independent of the time t, which proves proposition (1). Clearly this conclusion is actually a direct consequence of the fact that there is neither a characteristic time nor a relevant characteristic length in the problem.

Likewise, the pressure, density, temperature, and velocity of the flow behind the shock waves will be functions of ω_o, a_o, p_o, and t. In addition, they can be functions of the position x. The five variables ω_o, a_o, p_o, x, and t can be combined to give two independent nondimensional parameters, namely, $\frac{\omega_o}{a_o p_o}$ and $\frac{x}{a_o t}$. Therefore,

$$\frac{p(x,t)}{p_o} = F_2\left(\frac{\omega}{a_o p_o}, \frac{x}{a_o t}\right) \tag{32a}$$

$$\frac{u(x,t)}{a_o} = F_3\left(\frac{\omega}{a_o p_o}, \frac{x}{a_o t}\right) \tag{32b}$$

and so forth. In other words, the flow field must be "conical." Introducing a new independent variable $\zeta = \frac{x}{a_o t}$, it is a simple matter to reduce the governing partial differential equations (i.e., the continuity, momentum, and energy equations) to a system of ordinary differential equations with the independent variable ζ. It turns out that this system can be integrated without difficulty so that explicit solutions are obtained for the pressure, velocity, temperature, and density fields. The constants of integration are to be determined from the boundary conditions at the shock. However, a simpler approach is to assume a possible flow field which embodies all the features in propositions (1) to (6) and then to verify that it indeed satisfies all the conservation laws at all points.

For the purpose of formulation, suppose for the time being that the strength of the shock waves is known. Then the pressure p_1, velocity u_1, temperature T_1, and density ρ_1 immediately behind the shocks are also known. Since the shock strength is invariant with time, p_1, u_1, ρ_1, and T_1 will also be independent of time and are therefore constants. To satisfy proposition (2), assume

$$p(x,t) = p_1 \qquad \text{for} \quad |x| < V_s t \tag{33a}$$

where V_s is the shock speed. Proposition (3) states that

$$u(x,t) = \begin{cases} u_1 & \text{for} \quad 0 < x < V_s t \\ -u_1 & \text{for} \quad -V_s t < x < 0 \end{cases} \tag{33b}$$

As a consequence of this formula and proposition (4), the two contact surfaces which separate the hot and cool gas must be traveling with the velocity $\pm u_1$, respectively. (See fig. 2.) Proposition (5) then requires that

$$T(x,t) = \begin{cases} T_1 & \text{for} \quad u_1 t < |x| < V_s t \\ T^* & \text{for} \quad |x| < u_1 t \end{cases} \tag{33c}$$

where T^* is to be taken as a temperature tending to infinity. Proposition (6) requires that

$$\rho(x,t) = \begin{cases} \rho_1 & \text{for} \quad u_1 t < |x| < V_s t \\ \rho^* & \text{for} \quad |x| < u_1 t \end{cases} \tag{33d}$$

where ρ^* is to be taken as a density tending to zero in such a manner that $\rho^* T^* = \frac{p_1}{R}$.

It is clear that this solution satisfies all the conservation laws for any shock strength at any point in the flow field except perhaps at $x = 0$. For, if the flow field in the region $x > 0$ at any instant t is considered, it consists of a shock wave at $x = V_s t$, followed by a flow with uniform pressure and velocity consistent with the shock relations, and a contact surface at $x = u_1 t$, which moves with the fluid velocity. Such a flow field certainly satisfies all the conservation laws in the region $x > 0$. Hence, what must be examined is the question of whether the conservation theorems are also satisfied at $x = 0$. If they are satisfied there, then the conservation theorems for the system as a whole (i.e., over-all balance) will also be satisfied and the converse is also true. It is precisely this consideration that determines the strength of the shock wave.

Take a control surface consisting of the planes $x = \epsilon$, $x = -\epsilon$, and the segment of the wall of the tube between them. The conclusion will be the same whether ϵ is chosen to be an arbitrarily small number or a number large enough to include the entire disturbed field. It is found that the equations representing the conservation of mass and momentum are always identically satisfied no matter what is the shock strength, while the energy equation is satisfied if and only if the strength of the shock waves is such that the following condition

$$p_1 u_1 = \frac{\gamma - 1}{2\gamma} \omega_0 \tag{34}$$

is fulfilled. These statements can be proved easily if ϵ is chosen to be less than $V_s t$. When ϵ is taken to be larger than $V_s t$ the calculation is more involved, since use has to be made of the shock relations. It will be shown how the calculation proceeds for this more involved case (which incidently establishes the conservation laws for the system as a whole).

Take ϵ to be some fixed number greater than $V_s t$. Then at time t the shock waves will be at $x = V_s t$ and the contact surfaces, at $x = u_1 t$ (see fig. 3). The total mass inside the control surface consists of three parts: That between the two contact surfaces $2u_1 tA\rho^*$, which is negligible since ρ^* tends to zero eventually, that between the contact surface and the shock wave preceding it $2(V_s t - u_1 t)A\rho_1$, and that between the shock wave and the boundary of the control surface $2(\epsilon - V_s t)A\rho_0$. Thus, the total mass inside the control surface is

$$2u_1 At\rho^* + 2(V_s t - u_1 t)A\rho_1 + 2(\epsilon - V_s t)A\rho_0 = 2u_1 At\rho^* + 2\epsilon A\rho_0$$

where use has been made of the shock relation

$$\rho_1(V_s - u_1) = \rho_0 V_s \tag{35}$$

(i.e., the continuity equation at the shock). Now, since ρ^* is actually zero, the total mass contained in the control surface is invariant with time for all values of t and is equal to $2\epsilon A\rho_0$, the mass contained inside the control surface before heat is released to the medium. This demonstrates that the mass is conserved.

The momentum equation, being a vector equation, is automatically satisfied by virtue of the symmetry of flow with respect to the plane $x = 0$. Then consider the energy balance.

At the instant t, the energy inside the control surface also consists of three parts. It is given by

$$2u_1 tA\rho^*\left(C_v T^* + \frac{1}{2} u_1^2\right) + 2\left(V_s t - u_1 t\right)A\rho_1\left(C_v T_1 + \frac{1}{2} u_1^2\right) +$$

$$2\left(\epsilon - V_s t\right)A\rho_o C_v T_o = \frac{2}{\gamma - 1} p_1 u_1 tA + 2\epsilon A\rho_o C_v T_o +$$

$$2tAV_s\rho_o\left(C_v T_1 + \frac{1}{2} u_1^2 - C_v T_o\right)$$

since ρ^* tends to zero in such a manner that $\rho^* T^* = p_1/R$. Again use has been made of the continuity equation at the shock (35). The term $V_s\rho_o\left(C_v T_1 + \frac{1}{2} u_1^2 - C_v T_o\right)$ can be converted into $p_1 u_1$ if one makes use of the shock relations

$$C_p(T_1 - T_o) = V_s u_1 - \frac{1}{2} u_1^2 \tag{36}$$

$$p_1 - p_o = \rho_o V_s u_1 \tag{37}$$

$$\left(1 - \frac{\rho_o}{\rho_1}\right)V_s = u_1$$

which are nothing but the energy, momentum, and continuity equations at the shock. Thus, the total energy contained inside the control surface at the instant t is given by

$$\frac{2}{\gamma - 1} p_1 u_1 tA + 2\epsilon A\rho_o C_v T_o + 2tAp_1 u_1 = 2\epsilon A\rho_o C_v T_o + \frac{2\gamma}{\gamma - 1} p_1 u_1 tA$$

The increase of energy in the control surface from the instant t to the instant $t + \delta t$ is therefore $(2\gamma/\gamma - 1)p_1 u_1 A\delta t$ where δt is chosen so small that $V_s(t + \delta t)$ is still less than ϵ. But the total amount of heat released in time t is $\omega_o A\delta t$. Consequently, the energy will be conserved if and only if

$$p_1 u_1 = \frac{\gamma - 1}{2\gamma}\,\omega_o$$

which proves equation (34).

One is now in a position to relate the shock strength with the rate of heat released per unit area ω_o. It is well known that the drift velocity u_1 produced by a shock wave of strength p_1/p_o is

$$\frac{u_1}{a_o} = \frac{\frac{1}{\gamma}\left(\frac{p_1}{p_o} - 1\right)}{\sqrt{\frac{\gamma + 1}{2\gamma}\frac{p_1}{p_o} + \frac{\gamma - 1}{2\gamma}}} \tag{38}$$

Substituting equation (38) into equation (34), the following relation is found between the strength of the shock wave generated and the parameter of rate of heat release $\omega_o/a_o p_o$:

$$\frac{\omega_o}{a_o p_o} = \frac{\frac{2}{\gamma - 1}\frac{p_1}{p_o}\left(\frac{p_1}{p_o} - 1\right)}{\sqrt{\frac{\gamma + 1}{2\gamma}\frac{p_1}{p_o} + \frac{\gamma - 1}{2\gamma}}} \tag{39}$$

One can then solve the quartic equation for p_1/p_o in terms of $\omega_o/a_o p_o$. A plot of p_1/p_o versus $\omega_o/a_o p_o$ is given in figure 4. Obviously, this curve has been constructed by finding the values of $\omega_o/a_o p_o$ for a sequence of values of p_1/p_o. The results of this calculation are presented in the following table:

$\frac{p_1}{p_o}$	$\frac{\omega_o}{a_o p_o}$	$\frac{p_1}{p_o}$	$\frac{\omega_o}{a_o p_o}$
2	7.34	10	152.5
4	31.8	15	290.9
6	65.2	50	1,867
8	105.7		

It is interesting that for a very high rate of heat release the strength of the shock wave varies with $\omega_o/a_o p_o$ according to a two-thirds law:

$$\frac{p_1}{p_o} = 0.325\left(\frac{\omega_o}{a_o p_o}\right)^{2/3} \tag{40}$$

(The coefficient 0.325 corresponds to a value of γ of 1.4.) Once p_1/p_o is known as a function of $\omega_o/a_o p_o$, the complete flow field is defined and given by equation (33).

Application of the foregoing results to the approximate estimation of pressure waves generated by autoignition of a mixture in a tube is discussed in the section entitled "SOME APPLICATIONS OF THEORY."

THREE-DIMENSIONAL THEORY

In the present section, the linearized theory of pressure waves generated by heat addition in three dimensions will first be discussed briefly. Then the construction of an exact solution in three dimensions which may be useful in predicting the asymptotic strength of the shock wave generated by a closed flame front expanding uniformly will be discussed.

According to the linearized theory, the differential equation governing the pressure field due to a moderate rate of heat release is given by equation (8). The solution of equation (8) in an open space is well known:

$$\frac{\delta p(x,y,z,t)}{\gamma p_o} = \frac{1}{4\pi a_o^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{r'} \frac{\partial}{\partial t}\left[\frac{q\left(\xi,\eta,\zeta,t - \frac{r'}{a_o}\right)}{C_p T_o}\right] d\xi\, d\eta\, d\zeta \tag{41}$$

where $r' = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}$ is the distance between the field point (x,y,z) and the source point (ξ,η,ζ).

The special case in which heat is released at a given point in space, say the origin of the coordinate system, is of particular interest. This means that

$$q(x,y,z,t) = Q(t)\delta(x,y,z) \tag{42}$$

where $\delta(x,y,z)$ is the δ-function and can be considered as the limit, as $\epsilon \longrightarrow 0$, of the function

$$\delta(x,y,z) = \begin{cases} 0 & \text{for } x^2 + y^2 + z^2 > \epsilon^2 \\ \dfrac{1}{\frac{4}{3}\pi\epsilon^3} & \text{for } x^2 + y^2 + z^2 < \epsilon^2 \end{cases} \tag{43}$$

The function $Q(t)$ has the physical significance of being the heat release per unit time. Substituting equations (43) into equation (41) and evaluating the integral,

$$\frac{\delta p(x,y,z,t)}{\gamma p_o} = \frac{1}{4\pi a_o^2} \frac{1}{r} \frac{\partial}{\partial t}\left[\frac{Q\left(t - \frac{r}{a_o}\right)}{C_p T_o}\right] \tag{44}$$

where r is the distance $\sqrt{x^2 + y^2 + z^2}$.

A comparison of equations (21) and (44) shows that one of the essential differences between the one-dimensional and three-dimensional cases is that the pressure waves generated in the former case depend on the rate of heat release (per unit area) while those generated in the latter case depend upon the time rate of change of rate of heat release.

In practice, it is of some interest to know the pressure and velocity fields produced by the sudden addition of a finite amount of heat into the medium. Thus, the flow field produced by a spark discharge is of this nature, although actually a linearized theory will not be adequate to describe this phenomenon with precision. Assume the variation of rate of heat release with time as shown in figure 5. In other words, at $t = 0+$, the rate of heat release $Q(t)$ increases suddenly from zero to a very high value and then decreases again to zero in a short interval of time. The pressure waves generated, according to equation (44), will vary with the derivative of $Q(t)$ and will therefore consist of a very steep compression front followed immediately by an expansion-compression zone (fig. 6). The velocity field produced may be calculated from equation (5). The distribution of velocity field along any radial line is

$$u_r = \frac{1}{4\pi} \frac{1}{r^2} \frac{Q\left(t - \frac{r}{a_o}\right)}{C_p T_o} + \frac{1}{4\pi a_o} \frac{1}{r} \frac{\partial}{\partial t}\left[\frac{Q\left(t - \frac{r}{a_o}\right)}{C_p T_o}\right] \tag{45}$$

Thus, the velocity distribution in the immediate neighborhood of the origin behaves like an incompressible source field. Moreover, the radial distribution of the velocity there varies like Q (near field in fig. 6), while that at a large distance away bears the same relationship with the pressure as that existing in the theory of plane wave (far field in fig. 6).

Next examine the possibility of constructing an exact solution in three dimensions as was done before in one dimension. First, consider the case in which heat is released at a uniform rate of Q_o units of energy per second at the origin. As in the preceding section, the undisturbed medium can be characterized by its pressure p_o and velocity of sound a_o. Then the strength of the shock wave generated, measured in terms of the pressure ratio p_1/p_o across the shock wave, will, in general, be a function of Q_o, a_o, p_o, and time t. That is,

$$\frac{p_1}{p_o} = F(Q_o,a_o,p_o,t) \tag{46a}$$

Now the four variables Q_o, a_o, p_o, and t can only be combined into a single nondimensional parameter $Q_o/a_o^3p_ot^2$. Since equation (46a) must be dimensionally correct, one has

$$\frac{p_1}{p_o} = F\left(\frac{Q_o}{a_o^3p_ot^2}\right) \tag{46b}$$

Consequently, in this case the shock strength must be a function of time - a fact which complicates greatly the construction of an exact solution. Since it is physically apparent that increasing the rate of heat release will increase the strength of the shock wave generated, it is concluded from equation (46b) that the shock wave generated must decay with time. A little reflection reveals immediately that the basic reason why the shock strength should depend on t is the existence of a characteristic time $\sqrt{\frac{Q_o}{a_o^3p_o}}$ in the problem. It is also clear immediately that in order to produce a shock wave whose strength is invariant with time one must add heat to the medium according to the law

$$Q(t) = \alpha t^2 \tag{47}$$

where α now has the dimension energy/time^3. For, in this case, the rate at which heat is released will be characterized by α and the shock strength p_1/p_o should be a function of α, a_o, p_o, and t instead. But the four variables α, a_o, p_o, and t can only be combined into a single nondimensional parameter $\alpha/a_o^3p_o$ which does not contain t. Consequently, the shock strength p_1/p_o will be a function of $\alpha/p_oa_o^3$ but not of t; that is,

$$\frac{p_1}{p_o} = F\left(\frac{\alpha}{a_o^3p_o}\right) \tag{48}$$

The construction of an exact solution will be attempted for this case. The "parabolic law" (equation (47)) is actually of some practical interest. Thus, if a spherical flame propagates into the fresh gas with a constant speed, the rate of heat generated by combustion is proportional to t^2.

Moreover, if the flame speed is small, the heat may be thought of as being released at the center of the sphere.

For the present case, the pressure, velocity, density, and temperature field must be a function of α, a_o, p_o, r, and t. Just as in the derivation of equation (32), a dimensional reasoning leads to the result

$$\frac{p}{p_o} = p\left(\frac{\alpha}{{a_o}^3 p_o}, \frac{r}{a_o t}\right) \tag{49a}$$

$$\frac{u}{a_o} = u\left(\frac{\alpha}{{a_o}^3 p_o}, \frac{r}{a_o t}\right) \tag{49b}$$

and so forth, so that the flow field is again conical. It is also known that there must be a spherical contact surface behind the shock wave which separates the heated gas (i.e., the "fireball") from the cool gas surrounding it. If the position of this spherical contact surface at any instant t is denoted by its radius r_c, then, in general, $r_c = G(\alpha, a_o, p_o, t)$. In nondimensional form, this equation can be written as

$$\frac{r_c}{a_o t} = G\left(\frac{\alpha}{{a_o}^3 p_o}\right) \tag{50}$$

so that the contact surface must move out with a uniform velocity, or the fireball must expand at a uniform rate. Consequently, the flow field outside the contact surface will be exactly the same as that produced by a solid sphere expanding at a uniform rate of $a_o G\left(\frac{\alpha}{{a_o}^3 p_o}\right)$. But the air waves generated by a uniformly expanding sphere have been solved by Taylor (ref. 5)[4] so that the shock strength will be determined as soon as the rate of expansion of the fireball is related with the nondimensional heat-release parameter $\alpha/{a_o}^3 p_o$. To find this relation it is necessary to construct first a solution which is valid inside the fireball.

Except at the origin $r = 0$ where heat is released, the fundamental hydrodynamic equations governing the flow inside the fireball are

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[4]In fact, the dependence of pressure and velocity field on r and t in Taylor's solution satisfies precisely requirement (49).

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho u_r\right) = 0 \tag{51a}$$

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r}\right) = -\frac{\partial p}{\partial r} \tag{51b}$$

$$\frac{\partial}{\partial t}\rho\left(C_V T + \frac{1}{2}u_r^2\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \rho u_r\left(C_p T + \frac{1}{2}u_r^2\right)\right] = 0 \tag{51c}$$

$$p = \rho R T \tag{51d}$$

Just as in the preceding section, it is expected that the temperature T in the fireball tends to infinity, while the density ρ of the gas there tends to zero in such a manner that

$$\rho T = p/R \tag{52}$$

If this assumption is made, then continuity equation (51a) is automatically satisfied provided that u_r is finite. The momentum equation (51b) will then be satisfied if p is independent of r. But according to equation (49a), if p is independent of r it must also be independent of t. Hence, it is assumed that

$$p = p_c$$

where p_c is the pressure at the contact surface. Finally, since $\rho T = \frac{p_c}{R}$, the energy equation will be satisfied provided that $r^2 u_r$ is a function of t only. To satisfy equation (49b), it is necessary that $u_r = (\text{Constant})t^2/r^2$. In fact, if u_c is the velocity of the contact surface,

$$\frac{u_r}{u_c} = \left(\frac{t u_c}{r}\right)^2 \qquad \text{for} \quad r > 0$$

because, at $r = r_c$ $(= u_c t)$, u_r must be equal to u_c. (Note that u_r is indeed finite for all values of $r > 0$.) By symmetry, $u_r = 0$ at $r = 0$.

Summarizing the results, the flow field inside the contact surface is given by

$$p = p_c \tag{53a}$$

$$\left.\begin{aligned} \frac{u_r}{u_c} &= \left(\frac{tu_c}{r}\right)^2 \qquad \text{for} \quad r > 0 \\ u_r &= 0 \quad \text{at} \quad r = 0 \end{aligned}\right\} \tag{53b}$$

$$T = T^* \tag{53c}$$

$$\rho = \rho^* \tag{53d}$$

where T^* tends to ∞ and ρ^* tends to zero in such a manner than

$$\rho^* T^* = p_c/R \tag{54}$$

Note that inside the fireball the velocity distribution is like an incompressible source field (just as that indicated by the linearized solution, eq. (45)).

Up to now, the conservation laws have been satisfied at all points inside and outside of the fireball, except at the point $r = 0$ itself. An examination of the conservation laws at this point will enable one to relate the velocity of expansion of the fireball u_c and the pressure in the fireball p_c with the heat-release parameter α (see eq. (55)). If u_c is assumed to be known for the time being, Taylor's solution will then give the shock strength p_1/p_o and the flow field outside the fireball as well as the value of p_c. It will be shown that the correctly assumed u_c must be that which yields a p_c consistent with equation (55).

Consider a spherical control surface of radius ϵ about the origin. Take ϵ so small that the control surface lies completely inside the fireball. The continuity and energy equations formulated for this control surface are:

$$\frac{\partial}{\partial t}\left(\int_0^\epsilon \rho^* 4\pi r^2 \, dr\right) + \rho^* u_c \left(\frac{tu_c}{\epsilon}\right)^2 4\pi\epsilon^2 = 0$$

$$\frac{\partial}{\partial t}\int_0^\epsilon \rho^*\left(C_v T^* + \frac{1}{2}\, u_r^2\right) 4\pi r^2 \, dr + \rho^* u_c \left(\frac{tu_c}{\epsilon}\right)^2 \left[C_v T^* + \frac{1}{2}\, u_c^2 \left(\frac{tu_c}{\epsilon}\right)^4\right] 4\pi\epsilon^2 + p_c u_c \left(\frac{tu_c}{\epsilon}\right)^2 4\pi\epsilon^2 = \alpha t^2$$

The first equation is automatically satisfied since ρ^* ultimately tends to zero. The second equation can be rewritten as

$$u_c\left(\frac{tu_c}{\epsilon}\right)^2 \frac{1}{\gamma - 1} p_c 4\pi\epsilon^2 + p_c u_c\left(\frac{tu_c}{\epsilon}\right)^2 4\pi\epsilon^2 = \alpha t^2$$

so that it will be satisfied if and only if

$$\frac{4\pi\gamma}{\gamma - 1} p_c u_c^3 = \alpha \tag{55}$$

Finally, the momentum equation is automatically satisfied by virtue of symmetry. Consequently, all the conservation laws are satisfied everywhere provided that the flow field outside the fireball is described by Taylor's solution, that inside the fireball, by equation (53), and equation (55) is satisfied. It is often more convenient to use the nondimensional form $\frac{\alpha}{a_o^3 p_o} = \frac{4\pi\gamma}{\gamma - 1}\left(\frac{p_c}{p_o}\right)\left(\frac{u_c}{a_o}\right)^3$ of equation (55).

The table below is essentially a reproduction (in the notation of the present paper) of table I of Taylor's paper (ref. 5), except that one more column (the last column) has been added to give the corresponding values of $\alpha/a_o^3 p_o$ in accordance with equation (55). In the table M_c is the Mach number of the contact surface and r_1 is the radius of the shock wave. Other notations have already been introduced earlier.

(1)	(2)	(3)	(4)	(5)	(6)	(7)
M_c	$\frac{r_1}{r_c}$	$\frac{p_1}{p_o}$	$\frac{u_c}{a_o}$	$\frac{p_1}{p_c}$	$\frac{p_c}{p_o}$	$\frac{\alpha}{a_o^3 p_o}$
0	-----	------	-----	-----	------	0
.2	4.93	1.000	0.203	0.928	1.075	.392
.4	2.44	1.003	.410	.775	1.295	3.89
.5	1.950	1.050	.523	.750	1.400	8.73
.6	1.763	1.169	.638	.749	1.569	17.74
.7	1.503	1.365	.761	.755	1.808	34.7
.8	1.392	1.629	.891	.774	2.105	64.9
1.0	1.256	2.400	1.180	.811	2.959	212
1.2	1.182	3.59	1.520	.847	4.250	651
1.4	1.135	5.60	1.953	.887	6.32	2,050
1.6	1.103	9.06	2.560	.917	9.89	7,230
1.8	1.083	17.95	3.598	.92	19.7	40,000
2.1	1.060	∞	∞	.93	∞	∞

Column (7) of the table gives the heat-release parameter, while column (3) gives the strength of the shock wave produced. The dependence of the shock strength on the heat-release parameter $\alpha/a_o^3p_o$ is also shown in figure 7. Column (4) of the table gives the velocity of expansion of the fireball relative to the sound speed in the undisturbed medium. Column (6) gives the pressure inside the fireball. Column (2) gives the ratio of the radii of the spherical shock and the fireball.

SOME APPLICATIONS OF THEORY

Consider first some possible applications of equation (39). It is clear that in practice there are cases in which heat is released at a uniform rate in a limited region (or a narrow band) of a tube. If the axial dimension of this region is small compared with the length of the tube, equation (39) would be expected to give approximately the correct strength of the shock wave produced by the heat released. If the tube is of infinite length then equation (39) would be expected to give the asymptotic value of the strength of the shock wave developed, whatever is the size of the region at which heat is actually released (provided that this region is finite). A similar idea applies to some ignition problems. Consider a tube of infinite length containing some combustible mixture. Suppose that at $t = 0$, for one cause or another, ignition begins at one section, say at $x = 0$, of the tube. Thus, the mixture may have been ignited by a hot surface or a grid, or automatically because of the existence of a local high-temperature region. A flame is developed and the combustion will tend to spread out into the fresh gas (see fig. 8). If the flame spreads out at a uniform rate, the total amount of heat generated as the result of combustion will be linearly proportional to time t and the rate of heat release is therefore constant. Furthermore, if the flame spreads out at not too high a speed, compared with the velocity of sound in the surrounding medium, the strength of the shock wave generated would be expected to be approximately the same as if heat had been released at a constant rate at the section $x = 0$. In fact, one can calculate the strength of the shock waves generated. Thus, supposing that the two flame fronts propagate away from the ignition plane $x = 0$ with a constant speed S_t, (the transformation velocity), the total amount of heat generated in time t is $2\rho_1 S_t A\bar{Q}t$ where ρ_1 is the density of the medium behind the shock waves, A is the cross-sectional area of the tube, and $\bar{Q}$ is the heating value of the mixture (in energy per unit mass). It follows therefore that the rate of heat release per unit area is constant and given by

$$\omega_o = 2\rho_1 S_t \bar{Q} \tag{56}$$

and the strength of the shock wave generated can be solved from equation (39)

$$\frac{\dfrac{2}{\gamma - 1}\dfrac{p_1}{p_o}\left(\dfrac{p_1}{p_o} - 1\right)}{\sqrt{\dfrac{\gamma + 1}{2\gamma}\dfrac{p_1}{p_o} + \dfrac{\gamma - 1}{\gamma + 1}}} = 2\,\frac{\rho_1 S_t \bar{Q}}{a_o p_o} \tag{57}$$

where ρ_1 is related to p_1/p_o by the Rankine-Hugoniot relation

$$\frac{\rho_1}{\rho_o} = \frac{1 + \dfrac{\gamma + 1}{\gamma - 1}\dfrac{p_1}{p_o}}{\dfrac{\gamma + 1}{\gamma - 1} + \dfrac{p_1}{p_o}}$$

Naturally, one likes to know if this gives the correct answer. Fortunately, in the present case there is an independent method of calculating the strength of the shock waves generated. Note that since the flow field must be symmetric about the plane $x = 0$, the same flow field would have been produced if the tube had been closed off at $x = 0$ (fig. 9). Now a flame propagating away from the end of a tube must generate a shock wave of such a strength that the boundary condition at $x = 0$ is satisfied. Let the shock wave have the strength p_1/p_o where p_1 is the pressure immediately behind the shock. The shock will then induce a flow giving rise to a drift velocity (fig. 10)

$$u_1 = \frac{\dfrac{a_o}{\gamma}\left(\dfrac{p_1}{p_o} - 1\right)}{\sqrt{\dfrac{\gamma + 1}{2\gamma}\dfrac{p_1}{p_o} + \dfrac{\gamma - 1}{2\gamma}}} \tag{58}$$

where a_o is the velocity of sound in the medium ahead of the shock wave. Since the flame is assumed to be propagating with a constant speed S_t relative to the medium, it will be seen to propagate with an apparent speed of $u_1 + S_t$. Now the flame itself induces a flow behind it. From the viewpoint of an observer riding on the flame, the burned gas is leaving the flame with a velocity equal to $S_t \dfrac{\rho_1}{\rho_2}$ where ρ_1 is the density of the gas ahead of the flame (i.e., that behind the shock wave) and ρ_2 is that behind the flame. Hence, from the viewpoint of an observer in the laboratory (i.e., one who is fixed with respect to the undisturbed medium ahead of the shock waves), the burned gas will be moving away from the wall $x = 0$ with a speed equal to $u_1 + S_t - S_t \frac{\rho_1}{\rho_2}$. But the boundary condition at the wall dictates that the velocity of the flow at the fixed wall is zero. This condition is satisfied if $u_1 + S_t - S_t \frac{\rho_1}{\rho_2} = 0$ or

$$u_1 = S_t\left(\frac{\rho_1}{\rho_2} - 1\right) \tag{59}$$

which together with equation (59) determines the shock strength since ρ_1/ρ_2 is a fixed ratio once the mixture is specified. Now, if u_1 is small compared with the velocity of sound a_1 in the medium ahead of the flame, a simple consideration of momentum balance at the flame leads to the result[5]

$$p_1 = p_2 \tag{60}$$

Hence, by the gas law, $\rho_1/\rho_2 = T_2/T_1$, and equation (59) becomes

$$u_1 = S_t\left(\frac{T_2}{T_1} - 1\right) \tag{61}$$

But consideration of energy balance at the flame front shows that

$$\bar{Q} = C_p(T_2 - T_1) \tag{62}$$

provided that $\frac{S_t}{a_1} \ll 1$. Hence, equation (61) can be written simply as

$$u_1 = \frac{S_t\bar{Q}}{C_pT_1} \tag{63}$$

Substituting equation (63) into equation (58) and making use of the gas law, it is found that the shock strength p_1/p_o must satisfy the equation

$$\frac{\gamma - 1}{\gamma} \frac{S_t\bar{Q}}{p_1/\rho_1} = \frac{\frac{a_o}{\gamma}\left(\frac{p_1}{p_o} - 1\right)}{\sqrt{\frac{\gamma + 1}{2\gamma} \frac{p_1}{p_o} + \frac{\gamma - 1}{2\gamma}}} \tag{64}$$

[5]For proof of eqs. (60) and (62), see, e.g., ref. 6 or 7.

which after some simple reductions becomes identical to equation (57). Thus, it is seen that equation (57) indeed gives the correct value of the strength of the shock wave generated by combustion. The fact that in this case the shock strength can be predicted exactly by another method has not made the usefulness of equation (39) less, for, in the first place, it has strengthened confidence in its application to practical problems, and, secondly, equation (39) can be applied to many other cases (some of which have been mentioned at the beginning of this section) where no other simple means is available for estimating the approximate strength of the shock wave produced. Finally, the corresponding problem in three dimensions can be solved by the theory developed in the last section.

Mirels suggested that the condition under which the pressure wave produced by flame propagation can be found by considering an equal rate of heat release at a fixed station be considered. He noted that the equivalence of these two approaches is indicated for the planar case but is assumed for the three-dimensional case. (See the statement after equation (48): "If the flame speed is small, the heat may be thought of as being released at the center of the sphere.")

In response to this suggestion, the author would like to add that before any comparison can be made for the three-dimensional case the exact solution for the flow field generated by a uniformly expanding spherical flame must be known. This solution can, in fact, be constructed. It may be worth while to point out that, depending on the values of the flame speed and the heating value of the mixture, the flow field may assume qualitatively different natures. When the flame speed and heating value of the mixture are low enough (the present case), the flow field outside the flame front is similar to that generated by a uniformly expanding sphere, the speed of expansion of which is related to the flame speed and the heating value of the mixture. Inside the spherical flame the medium is at rest. When the flame speed and heating value of the mixture are high enough, the flow field inside the spherical flame is no longer entirely at rest. There is now a family of central-expansion spherical waves following immediately behind the flame front which is now propagating at the lower Chapman-Jouguet speed. The flow field outside of the flame front is still similar to that generated by a uniformly expanding sphere. When the flame speed and heating value of the mixture are extremely high, the flame front catches up with the shock wave to form a detonation front right after the mixture is ignited. This last case has been analyzed by Taylor (ref. 8) and independently by Doering and Burkhardt (ref. 9). However, all these exact solutions cannot be given in closed analytic forms which involve only the elementary transcendental functions. When the flame speed is small, the shock wave generated by the flame is <u>extremely</u> weak, mainly because of the fact that the shock wave is propagating into an open space. On the other hand, the equivalence of the flow field produced by the flame and a heat source releasing heat at an equivalent parabolic rate is true only when the flame speed is small. Consequently, it does not seem worth while from the practical point of view to look into this particular example in any greater detail.

When he was commenting on the equivalence of a flame front and a heater, the following argument was also advanced by Mirels: It can be shown for the planar case that the equivalence of the two approaches requires that the ratio of specific heats be the same for the burned and unburned gases and that the kinetic-energy terms be negligible. Take the case of a flame originating at $x = 0$ at time $t = 0$. The tx-diagram is indicated in figure 11(a). The equivalent problem, with heat addition at section $x = 0$, is indicated in figure 11(b). Conditions in region (1) of the figures are the same for both cases but conditions in region (2) differ. In addition, the extent of region (2) is greater in figure 11(a) than in figure 11(b) since the flame moves faster than does the contact surface. However, the energy per unit volume of the gas behind the shock (for a given shock strength) can be shown to be the same for regions (1) and (2) and for figures 11(a) and 11(b). Neglecting kinetic energy, the energy per unit volume is

$$E = \rho C_V T = \frac{p}{\gamma - 1}$$

Since p is constant behind the shock, the energy per unit volume is constant provided γ is the same for the burned and unburned gases. Therefore the details of the temperature and density distributions behind the shock are unimportant. The strength of the shock depends on the rate of heat addition and the two approaches are equivalent.

With regard to this discussion, the fact that a flame front and a heating element are dynamically equivalent[6] when the ratio of specific heats is the same for the burned and unburned gases and when the flame speed is small compared with the local sound speed was independently found by the author in his study of the mechanism of generation of pressure waves at a flame front (ref. 10). Actually, the statement is rigorously true only if there is a current of flow through the heating element with velocity equal to the flame speed. The rigorous demonstration will not be presented here.

[6]When the pressure and velocity fields produced in two systems are identical, the systems are said to be "dynamically equivalent." Note that the temperature and density fields produced in the two systems need not be the same.

CONCLUDING REMARKS

When the effect of heat conductivity is neglected, the flow field resulting from the addition of heat into a medium is caused by the volumetric expansion of the heated gas. When the rate of heat release is moderate, a "reduction theorem" can be derived which reduces the problem of heat addition in a single plane in a tube of constant cross section to a problem of piston motion in the tube (the plane of heat addition being perpendicular to the tube axis). The fictitious pistons in the theorem correspond in reality to the interfaces which separate the heated and unheated gases. The solution for the more general case of heat addition in a region inside the tube can be constructed by superposition.

The exact solution for the flow field produced by uniform heating in a plane at a constant rate is also given. In particular, the strength of the shock waves resulting from such heating is calculated in terms of the (constant) rate of heat release. The formula also gives the asymptotic strength of the shock waves resulting from heating at a constant rate a finite volume of gaseous medium inside an infinitely long tube of constant cross section.

When the heat is added into the medium at a single point at a rate proportional to the time squared, the heated gas expands at a uniform rate, much like a uniformly expanding sphere. Taylor's solution enables one to calculate the relation between the shock wave produced and the rate of heat release. This relation also represents asymptotically the strength of a pressure wave generated by a closed flame front in a combustible mixture expanding uniformly at a constant speed which is small compared with the local sound speed.

The Johns Hopkins University,
Baltimore, Md., December 11, 1953.

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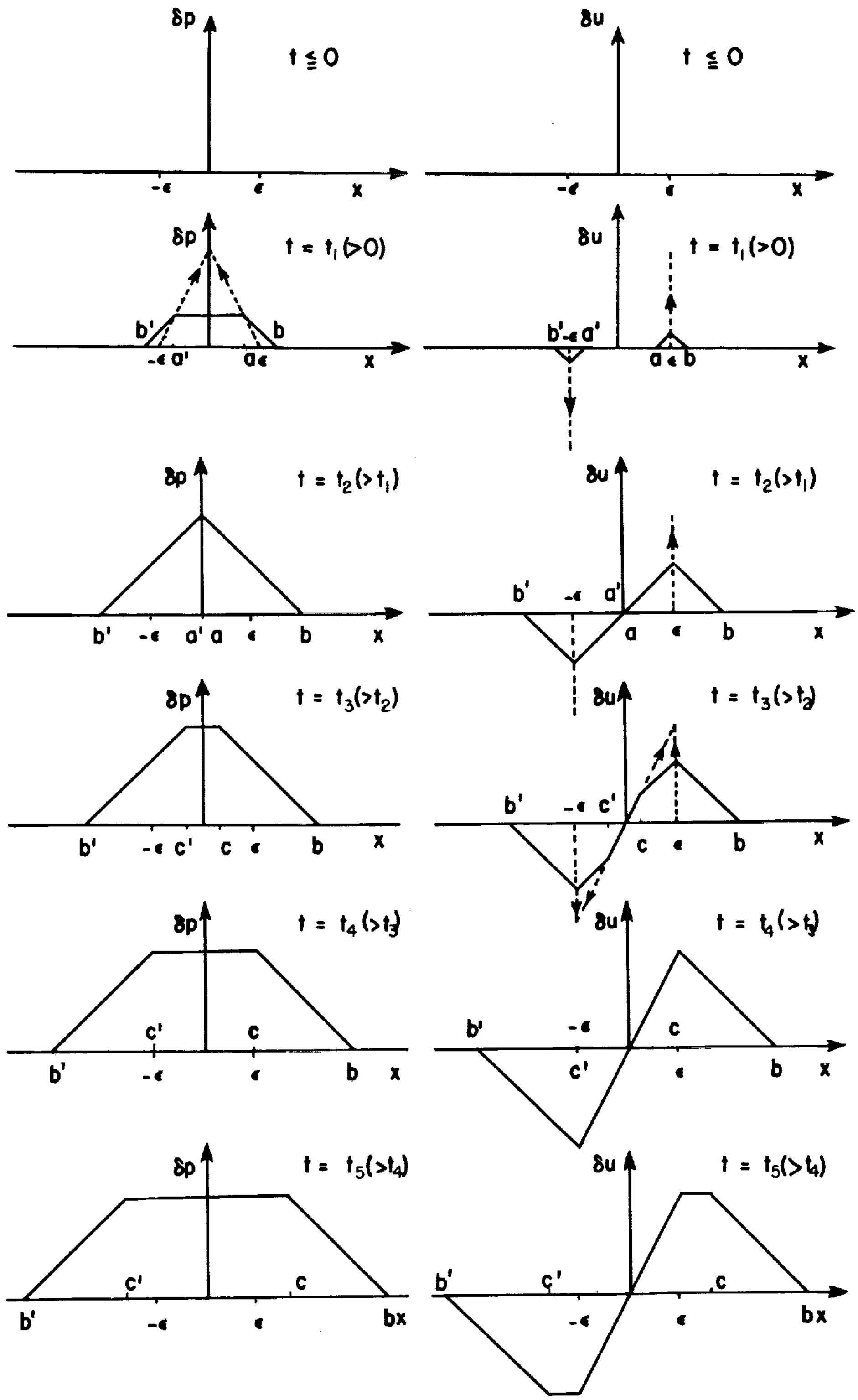


Figure 1.- Distribution of pressure and velocity produced by uniform heat addition in region $|x| \leqq \epsilon$.

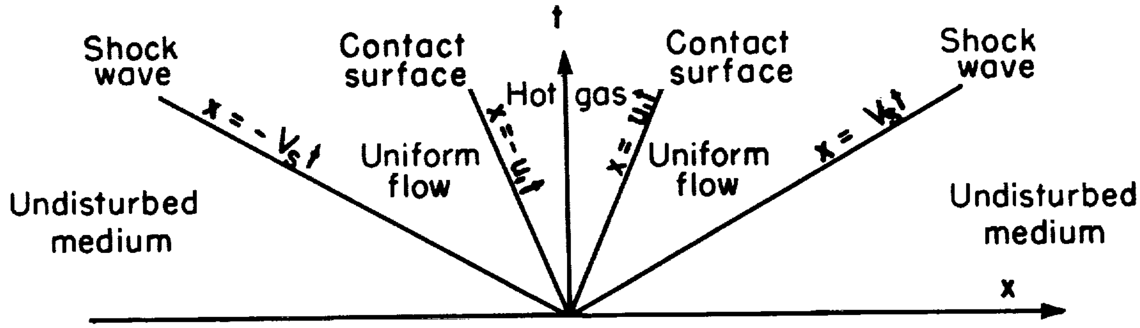


Figure 2.- Flow field in tx diagram.

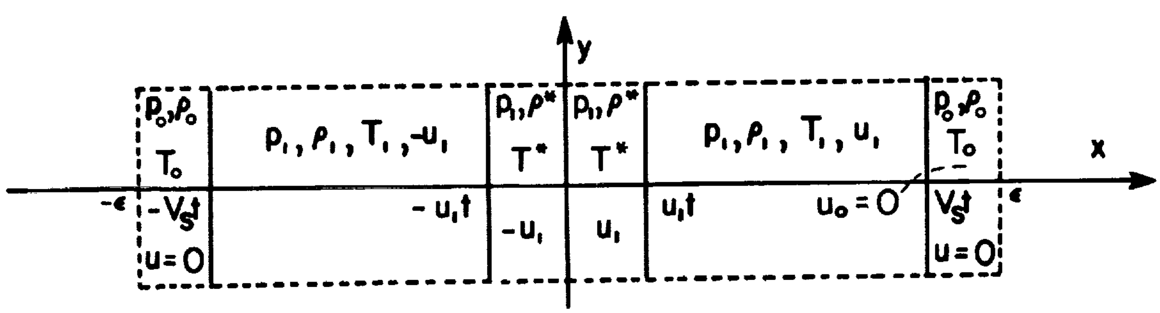


Figure 3.- Control surface.

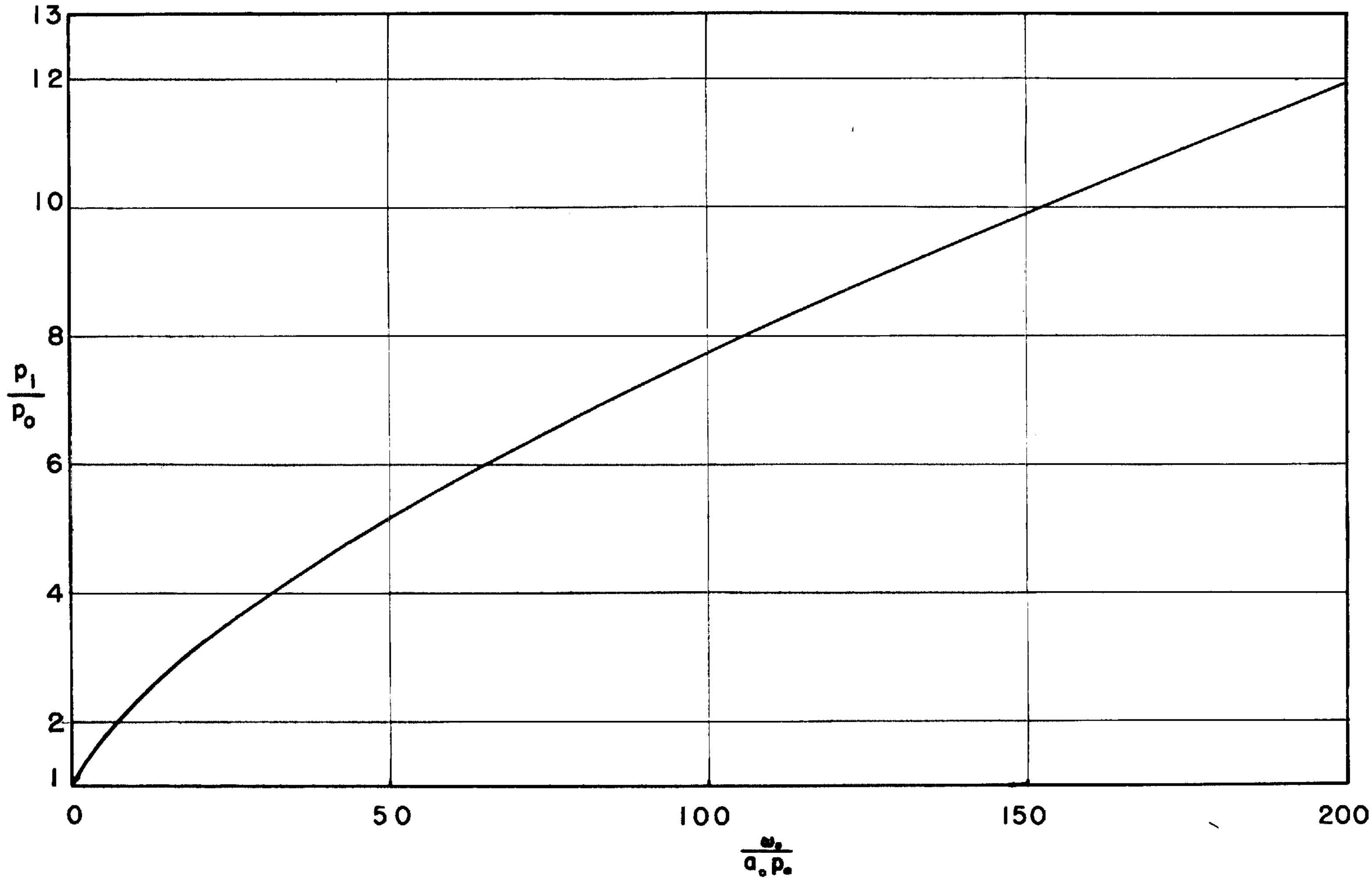


Figure 4.- Shock strength versus parameter of rate of heat release ω_0/a_0p_0.

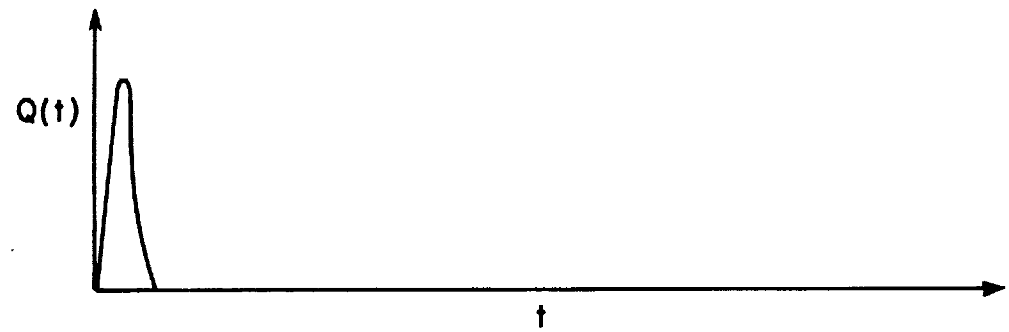
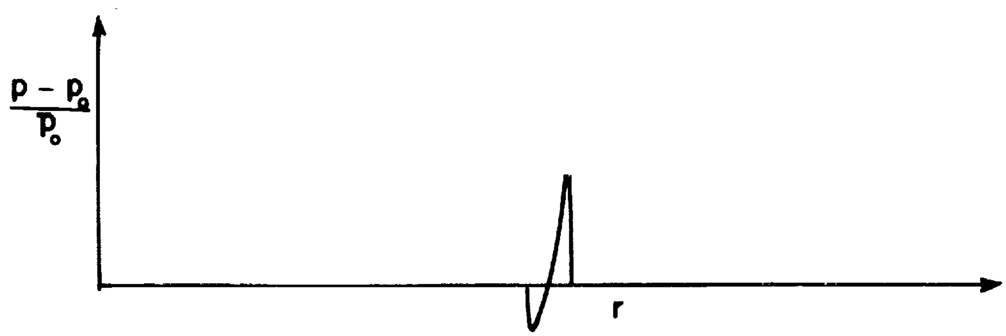
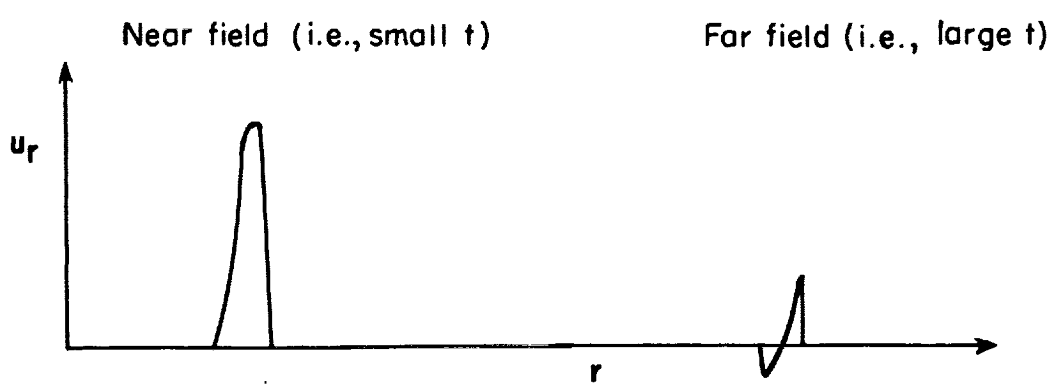


Figure 5.- Rate of heat release of a spark.



(a) Pressure field.



(b) Velocity field.

Figure 6.- Pressure and velocity fields produced by a spark.

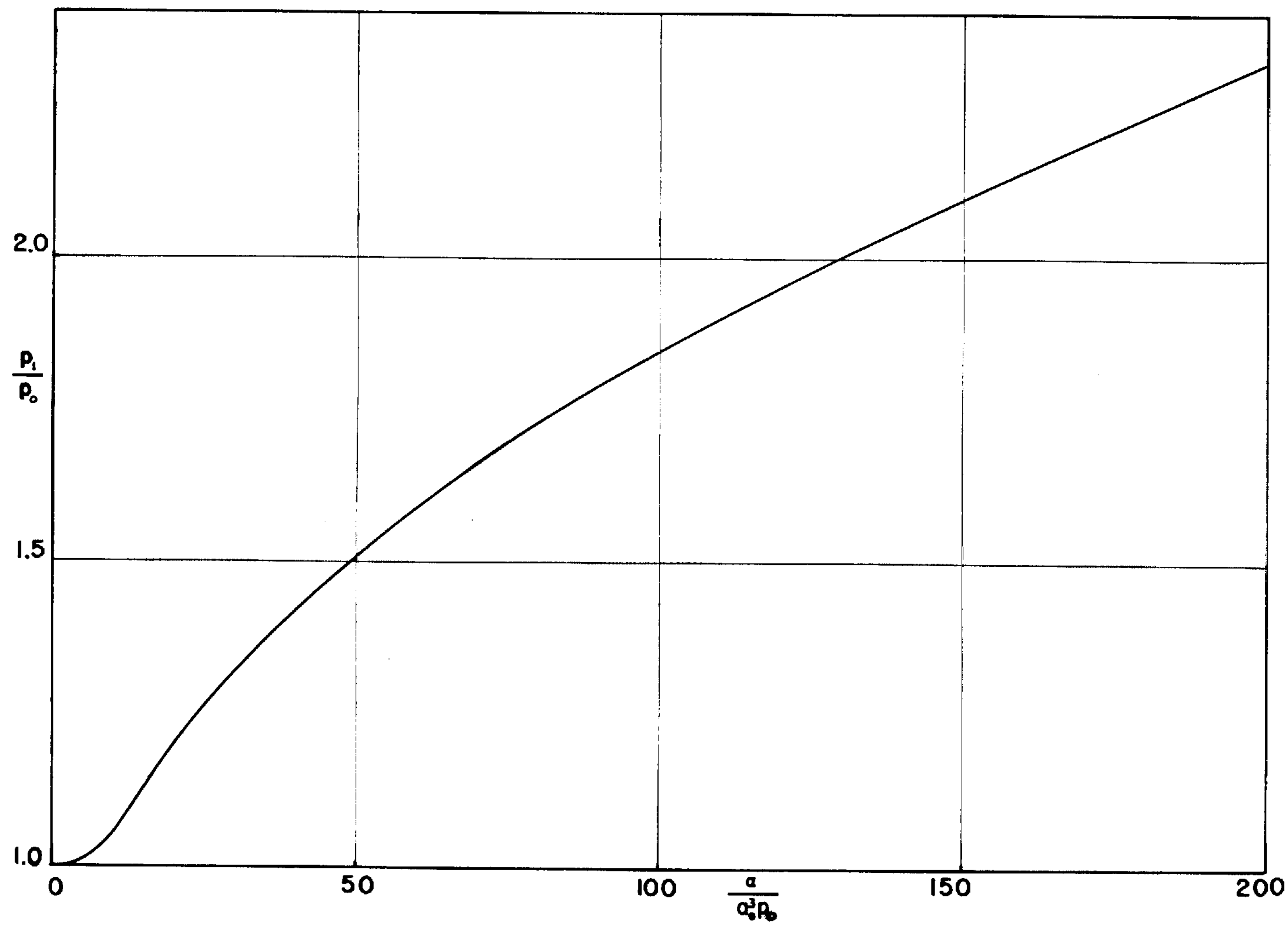


Figure 7.- Shock strength against parameter of rate of heat release $\alpha/a_0{}^3 p_0$.

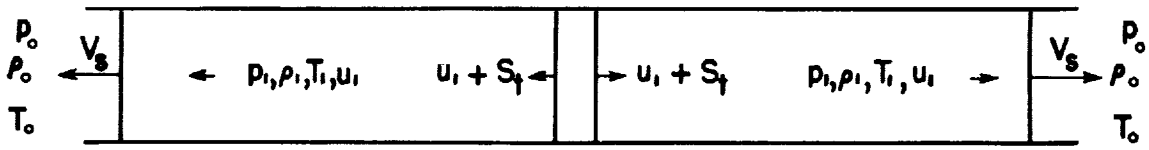


Figure 8.- Pressure waves generated as a result of ignition of a mixture in a tube.

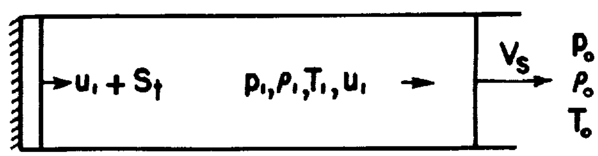


Figure 9.- Ignition of a mixture at a closed end of a tube.

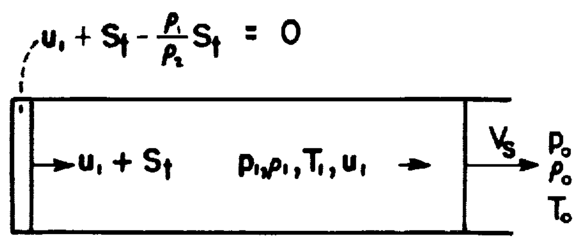
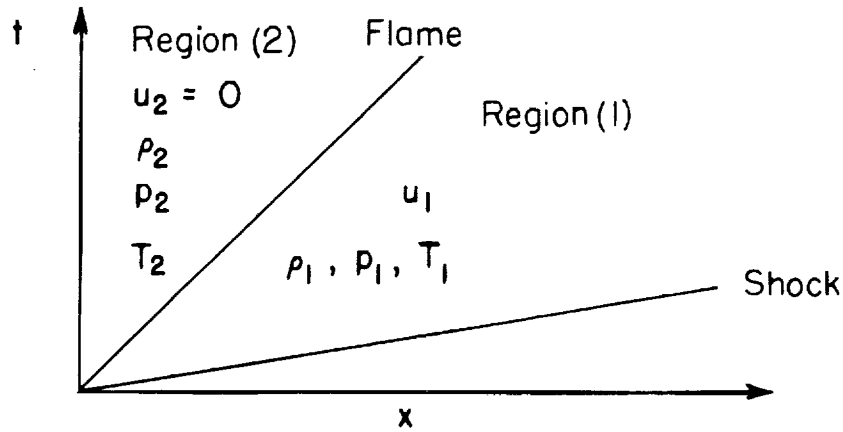
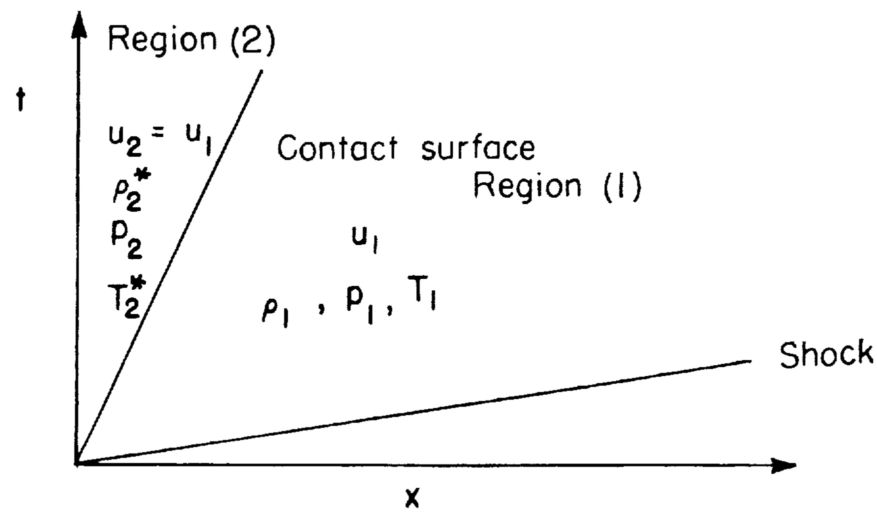


Figure 10.- Verification of equation (57) for a particular case.



(a) Flame propagating in tube.



(b) Heat addition at section $x = 0$.

Figure 11.- Case of flame originating at $x = 0$ and $t = 0$.